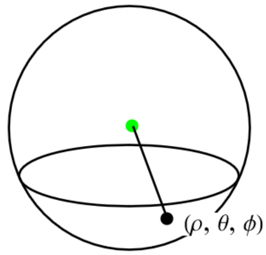
Exercise 19

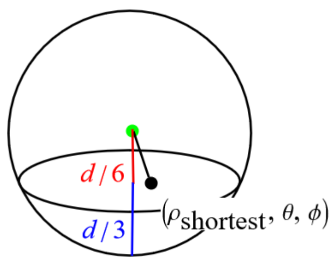
A vibrometer is to be designed that withstands the heating effects of its spherical enclosure of diameter d, which is buried to a depth $d/3$ in the earth, the upper portion being heated by the sun (assume the surface is flat). Heat conduction analysis requires a description of the buried portion of the enclosure in spherical coordinates. Find it.

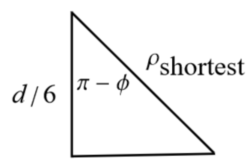
Solution

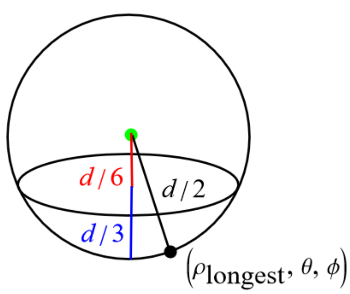
Consider a random point in the buried portion of the enclosure.



The points with the lowest and highest radii are shown below.







Use trigonometry to relate the sides of the triangle above.

$$\cos(\pi - \phi) = \frac{d/6}{\rho_{\text{shortest}}}$$
$$= \frac{d}{6\rho_{\text{shortest}}}$$

Solve for ρ_{shortest}.

$$\begin{aligned}\rho_{\text{shortest}} &= \frac{d}{6\cos(\pi-\phi)} \\ &= \frac{d}{6(\cos\pi\cos\phi+\sin\pi\sin\phi)} \\ &= \frac{d}{-6\cos\phi}\end{aligned}$$

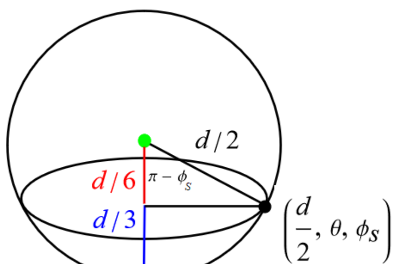
ρ_{longest} is just half the diameter. Therefore,

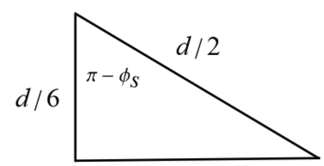
$$-\frac{d}{6\cos\phi} \leq \rho \leq \frac{d}{2}.$$

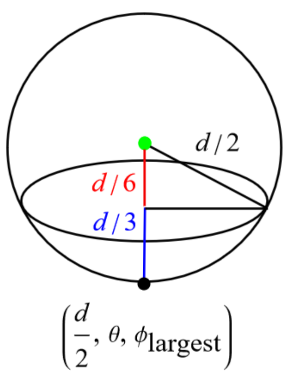
The azimuthal angle goes all the way around, so

$$0 \leq \theta \leq 2\pi.$$

Draw the points with the lowest and highest polar angles.







Use trigonometry to determine the shortest value of ϕ, ϕ_S.

$$\cos(\pi-\phi_S) = \frac{d/2}{d/6} \quad \rightarrow \quad \cos(\pi-\phi_S) = \frac{1}{3} \quad \rightarrow \quad \pi-\phi_S = \cos^{-1}\left(\frac{1}{3}\right)$$

Solve for ϕ_S.

$$\phi_S = \pi - \cos^{-1}\left(\frac{1}{3}\right)$$

The largest value of ϕ is just π. Therefore,

$$\pi - \cos^{-1}\left(\frac{1}{3}\right) \leq \phi \leq \pi.$$